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CONTINUOUS SELECTION THEOREMS ON PSEUDO SPACES WITH APPLICATIONS

Yen-Cherng Lin

Associate Professor, General Education Center, China Medical University

Abstract

In this paper, we first establish some upper semi-continuous selection theorems on pseudo spaces. As applications of our results, some fixed point theorems, coincidence theorems and collective fixed point theorems are established with much generalized convexity conditions on set-valued mappings with much simpler methods.

Key words: Pseudo spaces, Pseudo convex sets, Relative pseudo convex sets, Upper semi-continuous selections theorems, Coincidence theorems.

Requests for reprints should be sent to Lin, Yen-Cherng, General Education Center, China Medical University, 91 Hsueh-Shih Road, Taichung 404, Taiwan. Email：yclin@mail.cmu.edu.tw

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1. Introduction

Let X be a non-empty set. $P(X)$ denotes the power set of X and $|X|$ the cardinality of X. Let Δ^n denote the standard n-simplex $(e_1,\ldots,e_{n+1})$ in R^{n+1}, where e_i is the ith unit vector in R^{n+1} for $i=1,2,\ldots,n+1$. Let X, Y be two topological spaces, $F:X\to P(Y)$ be a set-valued mappings for a set X into $P(Y)$. Let $F^{-1}:Y\to P(X)$ be defined by $x\in F^{-1}(y)$ if and only if $y\in F(x)$. For $A\subset Y$, we denote $F^{-1}(A)=\{x\in X:F(x)\cap A\neq\varnothing\}$. If $f:X\to Y$ is an upper semi-continuous function such that $f(x)\subset F(x)$, we said that $f(x)$ is a *upper semi-continuous selection* of $F(x)$. The continuous selection results were first introduced by E. Michael[12] in 1956 with single-valued case. Recently, many authors discussed this property on many different spaces such as Hausdorff topological vector spaces (e.g.[1],[4],[7],[17]), H-spaces (e.g.[2],[3],[5],[6]), and G-convex spaces(e.g.[14],[15],[18]). The purpose of this paper is to establish upper semi-continuous selection theorems on the pseudo spaces. As applications, we derive the fixed point theorems of a collective set-valued mappings with new convex conceptions. We also derive coincidence theorems by using our new continuous selection results.

2. Preliminaries

Throughout this paper, all topological spaces in this paper are assumed to be Hausdorff. A triple $(X,D,\{q_A\})$ is said to be a *pseudo spaces* if X is a topological space, D be a nonempty set and for each nonempty finite subset A of D, there is a corresponding mapping $q_A:\Delta^{|A|-1}\to P(X)$ is an upper semi-continuous mapping with nonempty compact values such that the following two conditions hold: (1) there is an upper semi-continuous mapping $q_B:\Delta^{|B|-1}\to P(X)$ with nonempty compact values such that q_B is a restriction of q_A on $\Delta^{|B|-1}$ for all $\varnothing\neq B\subset A$ and (2) there is an upper semi-continuous mapping $q_C:\Delta^{|C|-1}\to P(X)$ with nonempty compact values such that q_A is a restriction of

q_C on $\Delta^{|A|-1}$ for all $A \subset C \subset D$.

If $D = X$, the triple $(X, D, \{q_A\})$ can be written by $(X, \{q_A\})$. An example of the pseudo H-space is given as follows.

Example 2.1. For any given G-convex space (X, D, Γ). Let Y be a topological space and $F : X \to P(Y)$ be upper semi-continuous with nonempty compact values. Then for each nonempty finite subset A of D, there is a continuous function $p_A : \Delta^{|A|-1} \to \Gamma_A$. Define $q_A := F \circ p_A : \Delta^{|A|-1} \to P(Y)$. Then q is upper semi-continuous with nonempty compact values. Therefore, $(Y, D, \{q_A\})$ forms a pseudo space.

A subset C of X is said to be *pseudo convex* if for each nonempty finite subset A of $C \cap D$, there is a $q_A : \Delta^{|A|-1} \to P(X)$, such that $q_A(\Delta^{|A|-1}) \subset C$.

Let $(X, D, \{q_A\})$ be a pseudo spaces, P be a nonempty finite subset of X and $Q \cap D \neq \emptyset$, we say that P is *pseudo convex relative to* Q if for each nonempty finite subset A of $Q \cap D$, there is a $q_A : \Delta^{|A|-1} \to P(X)$, such that $q_A(\Delta^{|A|-1}) \subset P$. We note that if $Q \cap D$ is non-empty and P is pseudo H-convex relative to Q, then P is automatically non-empty. In this convex sense, we don't know whether the sets on X, D are pseudo convex or not. Actually, we need not to discuss them in our context.

For topological spaces X, $A \subset X$, $int_X A$ denote the relative interior of A in X, we shall denote $int_X A$ by $intA$. A is said to be *compactly open* if for any compact set K in X, $A \cap K$ is open in K. For topological spaces X and Y, a set-valued map $F : X \to P(Y)$ is called

(1) *compact* if $\overline{F(X)}$ is compact in Y, where $\overline{B}$ denotes the closure of a set B.

(2) *transfer open*[16] if for every $x \in X$ and $y \in F(x)$ implies that there exists a point $x' \in X$ such that $y \in \operatorname{int} F(x')$.

The notation $F|_K$ means that F is restricted to K.

Lemma 2.1.[16] *Let* X *and* Y *be two topological spaces. Then* $F : X \to P(Y)$ *is transfer open if and only if* $\cup\{F(x) : x \in X\} = \cup\{intF(x) : x \in X\}$.

3. Upper Semi-continuous Selection Results

Now, we establish the following upper semi-continuous selection theorem which is the main result of this paper.

Theorem 3.1. *Let* X *be a paracompact topological space, the triple* $(Y, D, \{q_A\})$ *be a pseudo space,* $T : X \to P(Y)$, $S : X \to P(D)$ *be two set-valued mappings satisfying the following conditions:*

(i) *for each* $x \in X$, $T(x)$ *is pseudo convex relative to* $(intS^{-1})^{-1}(x)$*; and*

(ii) *there exists a nonempty finite subset* M *of* D *with* $|M| = n+1$ *for some* $n \in N$*, such that*

$$X = \cup_{y \in M} intS^{-1}(y).$$

Then there exist a nonempty subset $B(x) \subset M$ *for each* $x \in X$*, a continuous function* $\psi : X \to \Delta^n$ and a mapping $q_M : \Delta^{|M|-1} \to P(Y)$ *such that* $q_M(\Delta^{|B(x)|-1}) \subset T(x)$ *for all* $x \in X$ *and* $f = q_M \circ \psi$ *is an upper semi-continuous selection of* T.

Proof. By using the paracompactness of X and the conditions (i)-(ii), we can deduce the conclusion of the theorem from the definition of pseudo space.

Remarks:

(1) It is clear that if $T(x)$ is pseudo convex relative to $S(x)$ for each $x \in X$, then $T(x)$ is pseudo convex relative to $(intS^{-1})^{-1}(x)$ if $(intS^{-1})^{-1}(x) \neq \varnothing$ for each $x \in X$.

(2) The condition (ii) of Theorem 3.1 is satisfied if the following conditions

hold:

(a) $X = \cup_{y\in D} intS^{-1}(y)$; and

(b) there is a nonempty compact subset K of X such that X, $K \subset \cup_{y\in M} intS^{-1}(y)$ for some nonempty finite subset M of D.

The following corollary follows immediately from Theorem 3.1.

Corollary 3.2. *Let X be a compact space, the triple $(Y, D, \{q_A\})$ be a pseudo space, $S: X \to P(D)$ and $T: X \to P(Y)$ be two set-valued mappings satisfying the following conditions:*

(i) *for each $x \in X$, $T(x)$ is pseudo convex relative to $(intS^{-1})^{-1}(x)$; and*

(ii) $X = \cup_{y\in D} intS^{-1}(y)$.

Then there exist a nonempty subset A of D with $|A| = n+1$ for some $n \in N$, a nonempty subset $B(x)$ of A for each $x \in X$, a continuous function $\psi: X \to \Delta^n$ such that $q_A(\Delta^{|B(x)|-1}) \subset T(x)$ for all $x \in X$ and $f = q_A \circ \psi$ is an upper semi-continuous selection of T.

Corollary 3.3. *Let X be a paracompact topological space, Y be a topological vector space, D be a nonempty subset of Y and $T: X \to P(Y)$ and $S: X \to P(D)$ be two set-valued mappings satisfying the following conditions:*

(i) *for each $x \in X$, $co((intS^{-1})^{-1}(x)) \subset T(x)$;*

(ii) $X = \cup\{intS^{-1}(y): y \in M\}$ *for some nonempty finite subset M of D with $|M| = n+1$ for some $n \in N$.*

Then there exist a nonempty subset $B(x)$ of M for each $x \in X$, continuous functions $\psi: X \to \Delta^n$ and a linear continuous function $q_M: \Delta^n \to Y$ such that

$co(B(x)) \subset T(x)$ for all $x \in X$ and $f = q_M \circ \psi$ is an upper semi-continuous selection of T.

Proof. Let $M = \{a_1, a_2, ..., a_{n+1}\}$, define a linear function q by $q_M(e_i) = \{a_i\}$ for each $i \in \{1, 2, ..., n+1\}$, then $q_M(\Delta^{|A|-1}) = co(A)$ for each nonempty finite subset A of M and $q_M : \Delta^n \to P(Y)$ is an upper semi-continuous mapping with nonempty compact values. Then Corollary 3.3 follows from Theorem 3.1.

Theorem 3.4. *Let X be a topological space, the triple $(Y, D, \{q_A\})$ be a pseudo space, $T : X \to P(Y)$ and $S : X \to P(D)$. Suppose that $S^{-1} : D \to P(X)$ has transfer open or $S^{-}(y)$ compactly open for all $y \in D$. Let $F : Y \to P(X)$ be compact set-valued maps satisfying the following conditions:*

(i) *for each $x \in F(Y)$, $T(x)$ is pseudo convex relative to $(intS^{-1})^{-1}(x)$;*

(ii) $\overline{F(Y)} \subset S^{-1}(D)$.

Then there exist a nonempty finite subset A of D with $|A| = n+1$ for some $n \in N$, a nonempty subset $B(x)$ of A for each $x \in X$, a continuous function $\psi : F(Y) \to \Delta^n$ such that $q_A(\Delta^{|B(x)|-1}) \subset T(x)$ for all $x \in F(Y)$ and $f = q_A \circ \psi$ is an upper semi-continuous selection of T.

Proof. Since F is compact, $\overline{F(Y)}$ is compact. Let $K = \overline{F(Y)}$. Then

$$K = (\cup\{S^{-1}(y) : y \in D\}) \cap K.$$

If S^{-1} is transfer open. Then, by Lemma 2.1,

$$\cup\{S^{-1}(y) : y \in D\} = \cup\{intS^{-1}(y) : y \in D\}.$$

Therefore

$$K = \cup\{intS^{-1}(y) : y \in D\} \cap K = \cup\{int_K S^{-1}(y) : y \in D\}.$$

If $S^{-1}(y)$ is compactly open for each $y \in D$. Then $S^{-1}(y) \cap K$ is open in K. Hence

$$S^{-1}(y) \cap K = int_K(S^{-1}(y) \cap K) = int_K S^{-1}(y).$$

In any case, $K = \cup\{int_K(S^{-1}(y)) : y \in D\}$. Following the same argument as Theorem 3.1, we prove Theorem 3.4.

4. Applications to Fixed Point Theorems

As applications of the results of upper semi-continuous selections, we have the following fixed point theorems.

Theorem 4.1. *Let* $(X, D, \{q_A\})$ *be a pseudo space with* q_A *have acyclic values in* X*, the mappings* $S: X \to P(D)$, $T: X \to P(X)$ *satisfy the following conditions:*

(i) *for each* $x \in X$, $T(x)$ *is pseudo convex relative to* $(intS^{-1})^{-1}(x)$*; and*

(ii) *there exists a nonempty finite subset* M *of* D *with* $|M| = n+1$ *of* D *for some* $n \in N$*, such that* $X = \cup_{y \in M} intS^{-1}(y)$.

Then there exists an $\bar{x} \in X$ *such that* $\bar{x} \in T(\bar{x})$.

Proof. It follows from Theorem 3.1 that there exist nonempty finite subset $B(x) \subset M$, and a continuous function $\psi : X \to \Delta^n$ such that $q_M(\Delta^{|B(x)|-1}) \subset T(x)$ for all $x \in X$ and $f = q_M \circ \psi$ is an upper semi-continuous selection of T. Since $\psi \circ q_M : \Delta^n \to \Delta^n$, it follows from Lefschetz-type fixed point theorem for composites of acyclic maps that there exists an $\bar{u} \in \Delta^n$ such that $\bar{u} \in \psi \circ q_M(\bar{u})$. Let $\bar{x} \in q_M(\bar{u})$ with $\bar{u} = \psi(\bar{x})$, then $\bar{x} \in X$ and $\bar{x} \in q_M \circ \psi(\bar{x}) = f(\bar{x}) \subset T(\bar{x})$ and the conclusion follows.

Theorem 4.2. *Let I be a finite index set,* $\{(X_i, D_i, \{q_{A_i}\})\}_{i \in I}$ *be any family of pseudo spaces with* q_{A_i} *has acyclic values for* $i \in I$*. Let* $X = \prod_{i \in I} X_i$ *be equipped with product topology. For each* $i \in I$*, let* $T_i : X \to P(X_i)$ *and* $S_i : X \to P(D_i)$ *be set-valued maps satisfying the following conditions:*

(i) *for each* $x \in X$, $T_i(x)$ *is pseudo convex relative to* $(intS_i^{-1})^{-1}(x)$*;*

(ii) *there exists a compact subset* K *of* X *such that for each* $i \in I$ *and each nonempty finite subset* M_i *of* D_i *with* $|M_i| = n_i + 1$ *for some* $n_i \in N$*, there exists a compact pseudo convex subset* L_{M_i} *of* X_i *containing* M_i

such that X , $K \subset intS_i^{-1}(L_{M_i} \cap D_i)$ *; and*

(iii) $K = \cup_{y_i \in D_i}(intS_i^{-1}(y_i) \cap K)$.

Then for each $i \in I$, *there exist* M_i , *a compact pseudo convex subset* L_{M_i} *containing* M_i , *a finite subset* A_i *of* $L_{M_i} \cap D_i$ *with* $|A_i| = n_i + 1$ *for some* $n_i \in N$, *and a finite subset* $B_i(x)$ *of* A_i *for each* $x \in \Pi_{i \in I} L_{M_i}$, $\psi_i : \prod_{i \in I} L_{M_i} \to \Delta^{n_i}$ *such that* $q_{A_i}(\Delta^{|B_i(x)|-1}) \subset T_i(x)$ *for each* $x \in \prod_{i \in I} L_{M_i}$ *and* $f_i = q_{A_i} \circ \psi_i$ *is an upper semi-continuous selection of* $T_i|_{\prod_{i \in I} L_{M_i}}$.

Proof. By (iii), for each $i \in I$, there exists a nonempty finite subset M_i of D_i such that equation K $\subset \cup\{intS_i^{-1}(y_i) : y_i \in M_i\}$. By (ii), there exists a compact pseudo convex subset $L_{M_i} \subset X_i$ containing M_i such that

$$X, \ K \subset \cup\{intS_i^{-1}(y_i) : y_i \in L_{M_i} \cap D_i\}.$$

Let $M = \prod_{i \in I} M_i$ and $L_M = \prod_{i \in I} L_{M_i}$. Then L_M is a compact subset of X . By (1) and (2),

$$L_M = \cup\{int_{L_M} S_i^{-1}(y_i) : y_i \in L_{M_i} \cap D_i\}.$$

It is obvious that $(L_{M_i}, D_i \cap L_{M_i}, \{q_{A_i}\})$ forms a pseudo space. From (i), for each $x \in L_M$, $T_i(x)$ is pseudo convex relative to $(int_{L_M} S_i^{-1})^{-1}(x)$. By Corollary 3.2, for each $i \in I$, there is a nonempty finite subset A_i of $D_i \cap L_{M_i}$ with $|A_i| = n_i + 1$ for some $n_i \in N$, $B_i(x) \subset A_i$, $\psi_i : L_M \to \Delta^{n_i}$ and $q_i(\Delta^{|B_i(x)|-1}) \subset T_i(x)$ for all $x \in L_M$ and $f_i = q_i \circ \psi_i$ is an upper semi-continuous selection of $T_i|_{L_M}$.

Remark: The condition (iii) of Theorem 4.2 is satisfied if $X = \cup_{y_i \in D_i} intS_i^{-1}(y_i)$.

As a consequence of Theorem 4.2, we have the following collective fixed point theorem.

Theorem 4.3. *Let I be a finite index set,* $\{(X_i, D_i, \{q_{A_i}\})\}_{i\in I}$ *be any family of pseudo spaces with* q_{A_i} *has acyclic values for* $i \in I$. *Let* $X = \prod_{i\in I} X_i$ *be equipped with product topology. For each* $i \in I$, *let* $T_i : X \to P(X_i)$ *and* $S_i : X \to P(D_i)$ *be set-valued maps satisfying the following conditions:*

(i) *for each* $x \in X$, $T_i(x)$ *is pseudo convex relative to* $(intS_i^{-1})^{-1}(x)$;

(ii) *there exists a compact subset* K *of* X *such that for each* $i \in I$ *and each* M_i *is a nonempty finite subset of* D_i, *there exists a compact pseudo convex subset* L_{M_i} *of* X_i *containing* M_i *such that for each* $i \in I$, X, $K \subset intS_i^{-1}(L_{M_i} \cap D_i)$; *and*

(iii) $K = \cup_{y_i \in D_i}(intS_i^{-1}(y_i) \cap K)$.

Then there exists $\bar{x} = (\bar{x}_i)_{i\in I} \in X$ *such that* $\bar{x}_i \in T_i(\bar{x})$ *for all* $i \in I$.

Proof. It follows from Theorem 4.2 that for each $i \in I$, there exists a nonempty finite subset M_i of D_i, a compact pseudo convex subset L_{M_i} containing M_i, a finite subset A_i of $L_{M_i} \cap D_i$ with $|A_i| = n_i + 1$ for some $n_i \in N$ and a finite subset $B_i(x) \subset A_i$ for each $x \in \Pi_{i\in I} L_{M_i}$, continuous functions $\psi_i : \prod_{i\in I} L_{M_i} \to \Delta^{n_i}$ such that $q_{A_i}(\Delta^{|B_i(x)|-1}) \subset T_i(x)$ and $f_i = q_{A_i} \circ \psi_i$ is an upper semi-continuous selection of $T_i|_{\prod_{i\in I} L_{M_i}}$. For each $i \in I$, let E_i be the finite dimensional vector space containing Δ^{n_i}. Let $C = \prod_{i\in I} \Delta^{n_i}$. Then C is a compact convex subset of the locally convex Hausdorff topological vector space $E=\prod_{i\in I}E_i$. Let $q_A:C\to\prod_{i\in I}L_{M_i}$ be defined by $q_A(z)=(q_{A_i}(z_i))_{i\in I}$ for $z\in C$, where z_i is the i th projection of z and $A=\prod_{i\in I}A_i$. Let $\psi:\prod_{i\in I}L_{M_i}\to C$ be defined by $\psi(x)=(\psi_i(x))_{i\in I}$ for $x\in\prod_{i\in I}L_{M_i}$.

Since $\psi\circ q_A:C\to C$ is upper semi-continuous with acyclic values, by Lefschetz-type fixed point theorem that there exists $\bar{u}\in C$ such that $\bar{u}\in\psi\circ q_A(\bar{u})$.

Let $\bar{x}\in q_A(\bar{u})$ with $\bar{u}=\psi(\bar{x})$. Then $\bar{x}\in\prod_{i\in I}L_{M_i}\subset X$ and $\bar{x}\in q_A(\bar{u})=q_A\circ\psi(\bar{x})$. Let $\bar{x}=(\bar{x}_i)_{i\in I}$. Then $\bar{x}_i\in q_{A_i}\circ\psi_i(\bar{x})$. Therefore, $x_i\in q_{A_i}\circ\psi_i(\bar{x})\subset T_i(\bar{x})$ for all $i\in I$.

Remark: In Theorem 4.3, I can be any index set if q_{A_i} is assumed to have convex values instead of acyclic values. Theorem 4.3 is also a pseudo space version of partial results of Theorem 1[10], it also slight generalized Theorem 1[1] with much simple proof. For the particular cases of Theorem 4.3, we have the following theorem.

Theorem 4.4. *Let I be a finite index set,* $\{X_i\}_{i\in I}$ *be any family of topological vector spaces. Let* $X=\prod_{i\in I}X_i$ *be equipped with product topology. For each* $i\in I$ *, let* $T_i:X\to P(X_i)$ *and* $S_i:X\to P(X_i)$ *be set-valued maps satisfying the following conditions:*

(i) *for every* $x\in X$ *and* $i\in I$, $co((intS_i^{-1})^{-1}(x))\subset T_i(x)$ *;*

(ii) *there exists a compact subset* K *of* X *such that for each* $i\in I$ *, there exists a nonempty compact convex subset* L_i *of* X_i *such that for each* $x\in X$, K *and* $i\in I$ *, there exists a* $y_i\in L_i$ *such that* $x\in intS_i^{-1}(y_i)$ *; and*

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(iii) $K = \cup_{y_i \in X_i} (intS_i^{-1}(y_i) \cap K)$.

Then there exists $\bar{x} = (\bar{x}_i)_{i \in I} \in X$ *such that* $\bar{x}_i \in T_i(\bar{x})$ *for all* $i \in I$.

Proof. Fixed any $i \in I$. From (iii), there is a nonempty finite subset $M_i \subset X_i$ such that $K \subset \cup_{y_i \in M_i} intS^{-1}(y_i)$. By (ii), X, $K \subset \cup_{y_i \in L_i} intS_i^{-1}(y_i)$. Then $X = \cup_{y_i \in L_{M_i}} intS_i^{-1}(y_i)$, where $L_{M_i} = co(L_i \cup M_i)$ is a nonempty compact convex subset of X_i. Let $L_M = \prod_{i \in I} L_{M_i}$, then L_M is also a nonempty compact convex subset of X. Hence there is a nonempty subset N_i of L_{M_i} such that $L_M = \cup_{y_i \in N_i} int_{L_M} S_i^{-1}(y_i)$. By taking $X = K = L_M$ and $D_i = N_i = \{a_{i_1}, a_{i_2}, \ldots, a_{i_{n_i+1}}\}$.

Define a linear function $q_i : \Delta^{n_i} \to L_{M_i}$ with $q_i(e_j) = \{a_{i_j}\}$ for $j = 1, 2, \ldots, n_i + 1$. Then $(L_{M_i}, D_i, \{q_{A_i}\})$ form a pseudo space and we can easy deduce the conclusion of Theorem 4.4 from Theorem 4.3.

Remark: In Theorem 4.4 if condition (i) and (iii) are replaced by (i′) and (ii′), where

(i′) for every $x \in X$ and $i \in I$, co $S_i(x) \subseteq T(x)$; and

(ii′) $X = \cup_{y_i \in X_i}$ int $S_i^{-1}(y_i)$.

Then Theorem 4.4 is reduced to Theorem 1[1].

5. Applications to Coincidence Theorems

As applications of our upper semi-continuous selection theorems, we discuss the following coincidence theorems.

Theorem 5.1. *Let* X *be a paracompact topological space, the triple* $(Y, D, \{q_A\})$ *be a pseudo space. Let* $F \in U_c^k(Y, X)$ *[13] and* K *be a compact subset of* X *,* $T: X \to P(Y)$ *and* $S: X \to P(D)$ *be set-valued maps satisfying the following conditions:*

(i) *for each* $x \in X$ *,* $T(x)$ *is pseudo convex relative to* $(intS^{-1})^{-1}(x)$ *;*

(ii) *there exists a finite subset* M *of* D *such that*

$$X = \cup\{intS^{-1}(y) : y \in M\}.$$

Then there exist $\bar{x} \in X$ *and* $\bar{y} \in Y$ *such that* $\bar{x} \in F(\bar{y})$ *and* $\bar{y} \in T(\bar{x})$.

Proof. Let $|M| = n+1$ for some $n \in N$. It follows from Theorem 3.1, there exist $B(x) \subset M$ for each $x \in X$, continuous function $\psi: X \to \Delta^n$ such that $q_M(\Delta^{|B(x)|-1}) \subset T(x)$ for all $x \in X$ and $f = q_M \circ \psi$ is an upper semi-continuous selection of T. Let $G = \psi \circ F \circ q_M$. Then $G \in U_c^k(\Delta^n, \Delta^n)$. It follows from Corollary 2[9] that there exists a point $\bar{u} \in \Delta^n$ such that $\bar{u} \in G(\bar{u}) = \psi \circ F \circ q_M(\bar{u})$. Let $\bar{y} \in q_M(\bar{u})$ with $\bar{u} \in \psi \circ F(\bar{y})$ and $\bar{x} \in F(\bar{y})$ with $\bar{u} = \psi(\bar{x})$. Then $\bar{y} \in q_M(\bar{u}) = q_M \circ \psi(\bar{x}) \subset T(\bar{x})$.

Theorem 5.2. *Let* X *be a topological space, the triple* $(Y, D, \{q_A\})$ *be a pseudo space.* $T: X \to P(Y)$ *and* $S: X \to P(D)$. *Suppose that* $S^{-1}: D \to P(X)$ *is transfer open or* $S^{-1}(y)$ *is compactly open for all* $y \in D$. $F \in U_c^k(Y, X)$ *is a compact set-valued map satisfied the following conditions:*

(i) *for each* $x \in F(Y)$, $T(x)$ *is pseudo convex relative to* $(intS^{-1})^{-1}(x)$ *; and*

(ii) $\overline{F(Y)} \subset S^{-1}(D)$.

Then there exist $\bar{x} \in X$ *and* $\bar{y} \in Y$ *such that* $\bar{x} \in F(\bar{y})$ *and* $\bar{y} \in T(\bar{x})$.

Proof. Applying Theorem 3.5 and following the same arguments as in Theorem 5.1, we prove Theorem 5.2.

Corollary 5.3. *Let* X *be a topological space,* (Y, D) *be a convex space.*

$T: X \to P(Y)$ *and* $S: X \to P(D)$. *Suppose that* $S^{-1}: D \to P(X)$ *is transfer open or* $S^{-1}(y)$ *is compactly open for all* $y \in D$. *Let* $F \in U_c^k(Y, X)$ *be a compact set-valued map satisfied the following conditions:*

(i) *for each* $x \in F(Y)$, $co((intS^{-1})^{-1}(x)) \subset T(x)$; *and*

(ii) $\overline{F(Y)} \subset S^{-1}(D)$.

Then there exist $\bar{x} \in X$ *and* $\bar{y} \in Y$ *such that* $\bar{x} \in F(\bar{y})$ *and* $\bar{y} \in T(\bar{x})$.

Proof. It is clear that the conclusion of Corollary 5.3 follows from Theorem 5.2.

Remark: Corollary 5.3 improves Theorem 2[13].

We denote that $A_c(\Omega, \Xi)$ is the family of composites of acyclic maps from Ω to Ξ.

Theorem 5.4. *Let* X *be a paracompact topological space, the triple* $(Y, D, \{q_A\})$ *be a pseudo space with* q_A *have acyclic values in* Y. *Let* $F: Y \to P(X)$ *be an acyclic mapping and* K *be a compact subset of* X, $T: X \to P(Y)$ *and* $S: X \to P(D)$ *be set-valued maps satisfying the following conditions:*

(i) *for each* $x \in X$, $T(x)$ *is pseudo convex relative to* $(intS^{-1})^{-1}(x)$; *and*

(ii) $X = \cup_{y \in M} intS^{-1}(y)$ *for some nonempty finite subset* M *of* D.

Then there exist $\bar{x} \in X$ *and* $\bar{y} \in Y$ *such that* $\bar{x} \in F(\bar{y})$ *and* $\bar{y} \in T(\bar{x})$.

Proof. Let $|M| = n+1$ for some $n \in N$. It follows from Theorem 3.1, there exist $B(x) \subset M$ for each $x \in X$, continuous function $\psi: X \to \Delta^n$ such that $q_M(\Delta^{|B(x)|-1}) \subset T(x)$ for all $x \in X$ and $f = q_M \circ \psi$ is an upper semi-continuous selection of T. Let $G = \psi \circ F \circ q_M$. Then $G \in A_c(\Delta^n, \Delta^n)$. It follows from Lefschetz-type fixed point theorem for composites of acyclic maps that there exists a point $\bar{u} \in \Delta^n$ such that $\bar{u} \in G(\bar{u}) = \psi \circ F \circ q_M(\bar{u})$. Let $\bar{y} \in q_M(\bar{u})$ with $\bar{u} \in \psi \circ F(\bar{y})$ and $\bar{x} \in F(\bar{y})$ with $\bar{u} = \psi(\bar{x})$. Then $\bar{y} \in q_M(\bar{u}) = q_M \circ \psi(\bar{x}) \subset T(\bar{x})$.

Theorem 5.5. *Let* X *be a topological space, the triple* $(Y, D, \{q_A\})$ *be a pseudo space.* $T : X \to P(Y)$ *and* $S : X \to P(D)$. *Suppose that* $S^{-1} : D \to P(X)$ *is transfer open or* $S^{-1}(y)$ *is compactly open for all* $y \in D$. $F \in A_c(Y, X)$ *is a compact set-valued map satisfied the following conditions:*

(i) *for each* $x \in F(Y)$, $T(x)$ *is pseudo convex relative to* $(intS^{-1})^{-1}(x)$; *and*

(ii) $\overline{F(Y)} \subset S^{-1}(D)$.

Then there exist $\bar{x} \in X$ *and* $\bar{y} \in Y$ *such that* $\bar{x} \in F(\bar{y})$ *and* $\bar{y} \in T(\bar{x})$.

Proof. It following the same arguments as in Theorem 5.4, we prove Theorem 5.5.

The following result can be derived easily by using the technique of Theorem 4.4 and Theorem 5.5.

Corollary 5.6. *Let* X *be a topological space,* (Y, D) *be a convex space.* $T : X \to P(Y)$ *and* $S : X \to P(D)$. *Suppose that* $S^{-1} : D \to P(X)$ *is transfer open or* $S^{-1}(y)$ *is compactly open for all* $y \in D$. *Let* $F \in A_c(Y, X)$ *be a compact set-valued map satisfied the following conditions:*

(i) *for each* $x \in F(Y)$ *and* A *is a nonempty finite subset of* $(intS^{-1})^{-1}(x)$, $co(A) \subset T(x)$; *and*

(ii) $\overline{F(Y)} \subset S^{-1}(D)$.

Then there exist $\bar{x} \in X$ *and* $\bar{y} \in Y$ *such that* $\bar{x} \in F(\bar{y})$ *and* $\bar{y} \in T(\bar{x})$.

Remark: Corollary 5.6 improves Theorem 2[13].

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C.E.P.S.

擬空間的連續選擇定理及其應用

林炎成
中國醫藥大學通識教育中心 副教授

摘 要

在本文中，我們首先建立擬空間的上半連續選擇定理。作為本定理的應用，我們也運用集合值映射具較廣義的凸集條件及較簡捷的方法，獲得一些定點定理、疊合理論及集體的定點定理的結果。

關鍵詞：擬空間, 擬凸集合, 相對擬凸集合, 上半連續選擇定理, 疊合理論。